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Coordinate modelling for static axially symmetric electrovac metrics†

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Abstract. A new technique of coordinate modelling is introduced. It is shown that the static axially symmetric electrovac field equations are easier to solve in terms of coordinates which are specially adapted to the electrostatic equipotential contours of the system. Several new exact solutions are readily found by separating variables and by employing complex analytic functions. A framework for the construction of general solutions of the axially symmetric electrostatic vacuum field equations is presented and the Weyl solutions are shown to be trivially extractable.

1. Introduction

The standard use of symmetry in the search for exact solutions of the Einstein field equations is an old and familiar tool. The successful imposition of spherical symmetry by Schwarzschild, cylindrical symmetry by Einstein and Rosen and static axial symmetry by Weyl and Levi-Civita readily come to mind as examples in which very long, complicated, nonlinear partial differential equations of general relativity become relatively compact and very manageable. However, once the obvious symmetry techniques are exhausted, one must seek new tools to extend the search for exact solutions. While some methods of a rather complicated mathematical character do exist, in this paper we introduce a new straightforward approach. Although our application is confined to the static, axially symmetric electrovacuum equations, it would appear to be extendable to other types of problems as well.

Specifically, while the Einstein field equations for the axially symmetric electrostatic vacuum fields are greatly reduced in complexity by the Weyl form of the metric, the equations which relate g_{00} and Φ, the electrostatic potential, are still in general difficult to solve. Weyl (1917) determined a whole class of solutions which are generated by harmonic functions when the gravitational and electrostatic equipotential surfaces are assumed to coincide ($g_{00} = g_{00}(\Phi)$). (We will refer to these as 'Weyl solutions' or solutions of the 'Weyl form'.) However, when this special condition is removed, only a handful of solutions have been found and then only by transformation techniques (Bonnor 1966, 1979, Kramer and Neugebauer 1969, Gautreau and Hoffman 1970, Kinnersley 1974, Herlt 1978, Carminati 1981).

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In this paper, we change from the cylindrical polar coordinates ρ, z used by Weyl to a general orthogonal set x^1, x^2. We then specify that the constant x^1 coordinate surfaces coincide with the constant electrostatic potential surfaces, i.e. $\Phi = \Phi(x^1)$. In the classical picture, the orthogonal x^2 contours then follow the electric field lines. In other words, the coordinate system, from the outset, is forced to follow the full electrostatic character of the system and one is in a position to benefit from whatever symmetry Φ may display. While there is an immediate gain in that Φ, which involves only one coordinate, simplifies the differential equations, there is a price to be paid in that part of the complexity of the problem has been transferred into that of the construction of the coordinate system. However, this division of labour is entirely worthwhile because each part is manageable and several new solutions are readily found. Moreover, for one particular important class ($\alpha = 1$, α defined in § 2), compatible coordinate system candidates are readily found by extracting the real and imaginary parts of complex analytic functions. In addition, the new approach enables us to derive a framework for the construction of general solutions of the Einstein field equations for axially symmetric electrostatic fields in a vacuum. From the general framework, the Weyl solutions emerge trivially without any requirement of the Weyl insight that $\int e^{-w}\, d\Phi$ is harmonic.

In § 2, we develop the formalism for electrostatic conforming coordinates and present the field equations and the linked coordinate equations. In § 3, we treat the simply separable form for ρ in detail. The simply separable ρ is shown to be consistent only with the simply separable α and α is hence chosen to be 1 with no loss in generality. All of the non-Weyl solutions are determined for this case. The electrostatic potential is shown either to diverge or to have a bounded angular behaviour.

In § 4, for α separable (and hence chosen to be 1), we consider ρ to be in the form of a separable expansion and develop the field equations. For a special subclass where all constants of the expansion for e^{Ψ} and e^{H} (defined in § 2) are non-vanishing, the expansion in ρ has at most two linearly independent terms. This form is developed in detail in § 5 where four new multi-parameter solutions are given. In § 6, the special cases where Ψ' and H' are respectively zero are treated. Four additional solutions are given.

In the $\alpha = 1$ class, a simple approach for the determination of compatible coordinate candidates via complex analytic functions is developed in § 7, and a new solution of the non-separable form, which arose from this method, is given.

In § 8, we show that the field equations, expressed in the modelled coordinate contours, readily lend themselves to infinite-series expansions and the Weyl solutions are then trivially extracted. In § 9, we consider the symmetry transformations which map existing static solutions into new stationary solutions. One of our solutions so transformed is shown to have a NUT-like aspect. Some properties of other solutions are also considered. We offer some concluding remarks in § 10.

2. The formalism

Static axially symmetric fields outside matter can be described by the Weyl (1917) metric†

$$ds^2 = e^{w}\, dt^2 - e^{v-w}[d\rho^2 + dz^2] - \rho^2 e^{-w}\, d\phi^2 \tag{2.1}$$

† A simple computation using the Newman–Penrose formalism shows that all metrics of this form must be of Petrov type I or type D.

where w and v are functions of ρ and z. In the electrostatic vacuum, w and Φ satisfy the Einstein equations

$$\nabla^2 w \equiv w_{\rho\rho} + w_{zz} + w_\rho/\rho = 2\,\mathrm{e}^{-w}\nabla\Phi \cdot \nabla\Phi \tag{2.2}$$

$$\nabla^2\Phi = \nabla\Phi \cdot \nabla w \equiv \Phi_\rho w_\rho + \Phi_z w_z \tag{2.3}$$

where subscripts indicate partial differentiation with respect to the indicated variable. Once w and Φ are known, v can be determined by quadratures. While outwardly simple in appearance, equations (2.2) and (2.3) are nonlinear and strongly coupled in w and Φ.

At this point, we introduce the orthogonal electrostatic conforming coordinates x^1, x^2:

$$\rho \to \rho(x^1, x^2), \qquad z \to z(x^1, x^2) \tag{2.4}$$

with Φ and t unchanged. The Weyl metric, equation (2.1), becomes

$$\mathrm{d}s^2 = \mathrm{e}^w\,\mathrm{d}t^2 - \mathrm{e}^{v-w}[\hat{g}_{11}\,\mathrm{d}x^{12} + \hat{g}_{22}\,\mathrm{d}x^{22}] - \rho^2(x^1, x^2)\,\mathrm{e}^{-w}\,\mathrm{d}\phi^2 \tag{2.5}$$

where

$$\hat{g}_{11} \equiv \rho_1{}^2 + z_1{}^2, \qquad \hat{g}_{22} \equiv \rho_2{}^2 + z_2{}^2 \tag{2.6}$$

and subscripts 1 and 2 denote partial differentiation with respect to x^1, x^2 respectively. The orthogonality of both the (ρ, z) and (x^1, x^2) coordinate systems is expressed by

$$\rho_1\rho_2 + z_1 z_2 = 0, \qquad \alpha^2\rho_1 z_1 + \rho_2 z_2 = 0, \qquad \alpha^2 \equiv \hat{g}_{22}/\hat{g}_{11}. \tag{2.7}$$

In the new system, the flat three-space operators ∇ and ∇^2 of equations (2.2) and (2.3) (neglecting the differentiation with respect to ϕ) are

$$\begin{aligned} \nabla &= \frac{\hat{a}_1}{\sqrt{\hat{g}_{11}}}\frac{\partial}{\partial x^1} + \frac{\hat{a}_2}{\sqrt{\hat{g}_{22}}}\frac{\partial}{\partial x^2} \\ \nabla^2 &= \frac{1}{\sqrt{|\hat{g}|}}\left[\frac{\partial}{\partial x^1}\left(\frac{\sqrt{|\hat{g}|}}{\hat{g}_{11}}\frac{\partial}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{\sqrt{|\hat{g}|}}{\hat{g}_{22}}\frac{\partial}{\partial x^2}\right)\right] \end{aligned} \tag{2.8}$$

where $\hat{a}_1$ and $\hat{a}_2$ are unit vectors in the x^1 and x^2 directions and $\hat{g} = \rho^2\hat{g}_{11}\hat{g}_{22}$ is the determinant of the flat three-space metric. Since x^1 is chosen to follow the electrostatic contours, $\Phi = \Phi(x^1)$ and from equations (2.3) and (2.8),

$$\frac{\hat{g}_{11}}{\Phi'\sqrt{|\hat{g}|}}\frac{\partial}{\partial x^1}\left(\frac{\sqrt{|\hat{g}|}}{\hat{g}_{11}}\Phi'\right) = \frac{\partial w}{\partial x^1}. \tag{2.9}$$

This is easily integrated to yield

$$\mathrm{e}^w(x^1, x^2) = \rho(x^1, x^2)\alpha|\Phi'(x^1)|\,\mathrm{e}^{-H(x^2)}, \qquad \alpha > 0 \tag{2.10}$$

where H is a function of integration.

Due to the $\Phi(x^1)$ constraint, the choice of the form of the x^1, x^2 coordinates determines the form of the solution. Their relationship to ρ and z specifies this choice and hence it is appropriate at this stage to return to the constraint equations (2.7) which govern them. These are equivalent to

$$\rho_2 = \alpha z_1, \qquad z_2 = -\alpha\rho_1 \tag{2.11}$$

which are recognised as the Cauchy–Riemann conditions when $\alpha = 1$. They are integrable provided

$$(\alpha\rho_1)_1+(\rho_2/\alpha)_2=0, \tag{2.12}$$

which is the two-dimensional Laplace equation in arbitrary orthogonal curvilinear coordinates.

Using equations (2.8) and (2.10), the other field equation (2.2) can be expressed as

$$2\,\mathrm{e}^H|\Phi'|=\frac{\partial}{\partial x^1}\left(\frac{(\rho\alpha\Phi')_1}{\Phi'}\right)+\frac{\partial}{\partial x^2}\left(\frac{(\rho\alpha)_2}{\alpha^2}-\frac{\rho H'}{\alpha}\right) \tag{2.13}$$

or, using equation (2.12), in the more symmetrical form

$$[\rho(\alpha_1+\alpha\Psi')]_1+[(\rho/\alpha)(\alpha_2/\alpha-H')]_2=2\,\mathrm{e}^{\Psi+H} \tag{2.14}$$

where

$$\Psi\equiv\ln|\Phi'|. \tag{2.15}$$

Equations (2.12) and (2.13) or (2.14) determine all static axially symmetric electrovac solutions. Because of the linkage of the coordinates to Φ, they are particularly well adapted to the determination of new exact solutions by the technique of variable separations. As particular examples, we will examine the solutions when

$$\rho=a(x^1)b(x^2) \tag{2.16}$$

and†

$$\rho=a_1(x^1)b_1(x^2)+a_2(x^1)b_2(x^2),\qquad \alpha=c(x^1)d(x^2). \tag{2.17}$$

Firstly, regardless of the form of ρ, if α is simply separable as in (2.17), it is clear from (2.5) and (2.7) that a coordinate transformation

$$x^1=f(\bar{x}^1),\qquad x^2=g(\bar{x}^2) \tag{2.18}$$

can be invoked, preserving the equipotential contours, which makes $\alpha=1$ in the $\bar{x}^1$, $\bar{x}^2$ system. Thus, for the example of (2.17), we will actually perform the computation in the $\alpha=1$ gauge. Moreover, we will show that if ρ is simply separable as in (2.16), it follows that α must be separable. Thus, the ρ-separable example will also, for simplicity, be considered in the simplest gauge as well, $\alpha=1$, preserving all generality.

When $\alpha=1$, equations (2.11) and (2.12) become the familiar Cauchy–Riemann and harmonic equations

$$\rho_2=z_1,\qquad z_2=-\rho_1,\qquad \rho_{11}+\rho_{22}=0 \tag{2.19}$$

and equation (2.14) becomes

$$(\rho\Psi')_1-(\rho H')_2=2\,\mathrm{e}^{\Psi+H}. \tag{2.20}$$

Spherical polar coordinates, Erez–Rosen (1959) coordinates and other known systems belong to this class.

The simplicity of the $\alpha=1$ class presents us with a second approach for the determination of new exact solutions. This is due to the fact that equations (2.19) allow one easily to determine infinitely many x^1, x^2 systems which satisfy the coordinate requirements. One simply chooses any complex analytic function of $x^1+\mathrm{i}x^2$. The real and imaginary parts are respectively chosen as z and ρ, which, of course, satisfy

† Subscripts 1 and 2 on a and b are used to distinguish the functions rather than represent differentiation.

equations (2.19). One then seeks Ψ, H pairs which, with the given ρ, satisfy equation (2.20). This technique and a new solution will be developed and exhibited in § 7.

3. The simply separable form

We first solve the ρ form of equation (2.16). This ρ, in conjunction with the first of equations (2.7) and the new variables

$$\bar{x}^1 \equiv \tfrac{1}{2}a^2, \qquad \bar{x}^2 \equiv \tfrac{1}{2}b^2 \tag{3.1}$$

yields†

$$(\partial z/\partial \bar{x}^1)(\partial z/\partial \bar{x}^2) = -1. \tag{3.2}$$

Using Charpit's method, the general integral of equation (3.2) is

$$z = ka^2/2 - b^2/2k + l \tag{3.3}$$

where k and l are constants. From (3.3), (2.16), (2.11) and (3.3), we see that α must be separable. Moreover, from the discussion of § 2, there is then no loss in generality in taking $\alpha = 1$, which we shall do. From (2.19) and (2.16),

$$a'' = \lambda^2 a, \qquad b'' = -\lambda^2 b \tag{3.4}$$

where the separation constant λ^2 has been chosen to be positive so that x^1 has a 'radial-like' character. Using the field equation (2.20), we see that there are two cases to consider:

$$\text{(i)}\ \mathrm{e}^H = kb, \qquad \text{(ii)}\ \mathrm{e}^\Psi = ka \tag{3.5}$$

where k is a constant. If (i) holds, then, using (2.10), we find that $w = w(x^1)$ and hence this is a member of the Weyl class. However, case (ii) yields the non-Weyl solutions

$$kb\, \mathrm{e}^H = \mathrm{sech}^2\, G \tag{3.6}$$

where‡

$$G = \begin{cases} -N^{-1}\tanh^{-1}[(1-\gamma^2 b^2)^{1/2}] + D, & \lambda \neq 0 \\ N^{-1}\ln b + D, & \lambda = 0 \end{cases} \tag{3.7}$$

D, N are constants and $\gamma^2 \equiv \lambda^2/N^2$.

From equations (3.5(ii)) and (3.4),

$$\Phi = \begin{cases} \pm k/\lambda^2(\lambda^2 a^2 + Q)^{1/2}, & \lambda \neq 0 \\ Ma^2, & \lambda = 0 \end{cases} \tag{3.8}$$

where Q and M are constants and§

$$\mathrm{e}^w = (kab\cosh G)^2 \tag{3.9}$$

follows from equations (2.16), (3.5(ii)) and (3.6). Finally, the connection of a and b with the cylindrical polar coordinates is completed by integrating the Cauchy–Riemann

† Note that for $a'b' = 0$, one is restricted to the contours of the cylindrical polar coordinates themselves.
‡ Note that for $\lambda \neq 0$, $N = -1$ and $D = 0$ gives a Weyl-class solution.
§ This is included in the Gautreau–Hoffman (1970) solutions.

equations:

$$(\lambda^2 z - L)^2 = (N^2 - \lambda^2 b^2)(Q + \lambda^2 a^2), \qquad \lambda \neq 0$$
$$2(z - L) = Qa^2 - b^2/Q, \qquad \lambda = 0 \tag{3.10}$$

where L is a constant of integration.

The possible asymptotic forms for Φ are

$$\Phi \sim \pm r \qquad \text{and} \qquad \Phi \sim \pm(Q^{1/2}/\lambda^2)\sin\theta \tag{3.11}$$

for the $\lambda \neq 0$ case and

$$\Phi \sim r[1 + 3\cos^2\theta]^{1/2} \qquad \text{or} \qquad r\sin\theta \tag{3.12}$$

for the $\lambda = 0$ case. It is interesting to note that in the second of equations (3.11), Φ approaches different finite limits in the different directions. When the latter is mapped into stationary vacuum form in § 9 using the Bonnor (1966) transformation we will see that the new metric displays the NUT-like (Newman *et al* 1963) behaviour insofar as spin is concerned.

4. Separable expansions

It is natural to develop a framework which fully utilises the single variable dependence of Φ. To this end, with α separable (and hence chosen to be unity), we assume a ρ of the form of a separable expansion

$$\rho = \sum_{j=1}^{n} a_j(x^1) b_j(x^2) \tag{4.1}$$

where the a_j and b_j are all linearly independent. Moreover, we assume for now that $\Psi' \neq 0$, $H' \neq 0$. From (4.1), and (2.20),

$$\sum_{j=1}^{n} [b_j(a_j\Psi')' - a_j(b_j H')'] = 2\,\mathrm{e}^{\Psi+H}. \tag{4.2}$$

In succession, equation (4.2) is divided by $a_1 b_1$, $(a_2/a_1)'(b_2/b_1)'$, $[(a_3/a_1)'/(a_2/a_1)'] \cdot [(b_3/b_1)'/(b_2/b_1)']$, etc and is differentiated with respect to x^1 and x^2 at each step until the series is exhausted. The final equation is then integrated back to yield

$$\mathrm{e}^{\Psi} = \sum_{j=1}^{n} l_j a_j \tag{4.3}$$

or

$$\mathrm{e}^{H} = \sum_{j=1}^{n} m_j b_j \tag{4.4}$$

where l_j, m_j are arbitrary constants.

Note that $\rho \to \rho$ under the scale transformation $a_j \to a_j/k_j$, $b_j \to k_j b_j$ (j not summed) for constants k_j. This can be used to make all of the non-vanishing l_j or m_j equal (not both simultaneously) so either $l_j = l$ or $m_j = m$. Next, equation (4.3) with $l_j = l$ or equation (4.4) with $m_j = m$ is substituted into each successive integrability equation, simplifying each in turn. For the case where either all the l_j's or all the m_j's are non-zero, one can exploit the invariance of ρ and e^Ψ or e^H under $a_j \leftrightarrow a_k$, $b_j \leftrightarrow b_k$, $j, k = 1, \ldots, n$ to obtain the following two distinct sets of coupled, nonlinear ordinary differential equations which express the integrability requirements:

I:
$$2\,e^\Psi = E_1 \sum_{k=1}^{n} a_k \tag{4.5}$$

$$(a_k \Psi')' = Ea_k + F \sum_{\substack{j=1\\ j\neq k}}^{n} a_j \tag{4.6}$$

$$(b_k H')' = Eb_k + F \sum_{\substack{j=1\\ j\neq k}}^{n} b_j - E_1\, e^H \tag{4.7}$$

II:
$$2\,e^H = E_1 \sum_{k=1}^{n} b_k \tag{4.8}$$

$$(b_k H')' = Eb_k + F \sum_{\substack{j=1\\ j\neq k}}^{n} b_j \tag{4.9}$$

$$(a_k \Psi')' = Ea_k + F \sum_{\substack{j=1\\ j\neq k}}^{n} a_j + E_1\, e^\Psi \tag{4.10}$$

$k = 1, 2, \ldots, n$; E, E_1, F are constants.

Note that equations (2.19), (2.20) and (4.1) are invariant under the transformation

$$\begin{aligned} &x^1 \to \mathrm{i}x^2, \qquad x^2 \to -\mathrm{i}x^1, \qquad \mathrm{i} = \sqrt{-1} \\ &a_k \leftrightarrow b_k, \qquad \Psi \leftrightarrow H \end{aligned} \tag{4.11}$$

which also links sets I and II. It is interesting to note that using either set I or set II, it can be shown that the expansion in ρ of (4.1) has at most two linearly independent terms. However, it must be stressed that this result follows only when either all the l_j's or all the m_j's in (4.3), (4.4) are non-zero. For this case, we now consider the form of ρ with two separable products of equation (2.17) when $\alpha = 1$.

5. Two separable products

For the ρ form of equation (2.17) in the $\alpha = 1$ gauge, it is most convenient to define

$$\begin{aligned} &\sqrt{2}\, A_1 \equiv a_1 + a_2, \qquad \sqrt{2}\, A_2 \equiv a_1 - a_2 \\ &\sqrt{2}\, B_1 \equiv b_1 + b_2, \qquad \sqrt{2}\, B_2 \equiv b_1 - b_2. \end{aligned} \tag{5.1}$$

From equations (5.1) and (2.17), ρ retains the original structure

$$\rho = A_1(x^1)B_1(x^2) + A_2(x^1)B_2(x^2), \qquad \alpha = 1. \tag{5.2}$$

The problem at hand is one of solving set I (equations (4.5)–(4.7)) or set II (equations (4.8)–(4.10)) in conjunction with this ρ which in turn must be harmonic, $\nabla^2\rho = 0$.

Until the end of this section, we will concentrate on set I. After some relabelling of constants, set I with equations (5.1) and (5.2) yield

$$2\,\mathrm{e}^{\Psi}=QA_1, \qquad A_1''=\beta^2 A_1 \tag{5.3), (5.4}$$

set I: $$[A_2A_1'/A_1]'=\sigma A_2, \qquad [B_1H']'=\beta^2B_1-Q\,\mathrm{e}^{H}, \qquad [B_2H']'=\sigma B_2. \tag{5.5), (5.6), (5.7}$$

The harmonic condition, $\nabla^2\rho=0$, in conjunction with equation (5.4), yields the following choice of constraint equations (depending on the selection of constants):

C.I(1) $$A_2''=k_1A_2+A_1, \qquad -B_1''=\beta^2B_1+B_2, \qquad -B_2''=k_1B_2 \tag{5.8}$$

or

C.I(2) $$A_2''=k_1A_2, \qquad B_1''=-\beta^2B_1, \qquad B_2''=-k_1B_2 \tag{5.9}$$

where β, σ and k_1 are constants.

From equations (5.4) and (5.5), with $\beta\neq 0$,†

$$A_1=D_1\cosh\xi, \qquad A_2=D_2\cosh\xi\,\sinh^{\delta-1}\xi \tag{5.10}$$

where $\xi\equiv\beta x^1+\tau$, $\delta\equiv\sigma/\beta^2$ and D_1, D_2 and τ are constants. The case where A_1 is of the form $\mathrm{e}^{\beta x^1}$ will be treated later when the constraint set C.I(2) is used. For the constraint set C.I(1), however, we find with equation (5.5) that the simple exponential solution leads to A_1 being proportional to A_2 which is the case already treated in § 3. Moreover, C.I(1) and equation (5.10) yield only two possibilities, $\delta=1$ or $\delta=3$. The $\delta=1$ trial again leads to a simply separable ρ and the $\delta=3$ trial leads to a form for e^H, B_1 and B_2 which cannot satisfy equation (5.6). Therefore C.I(1) leads to no new solutions.

Considering C.I(2), equations (5.10) yield the compatibility condition

$$(\delta^2-k_1/\beta^2)+(\delta-1)(\delta-2)\sinh^{-2}\xi=0. \tag{5.11}$$

$\delta=1$ again gives the simply separable ρ. The $\delta=2$ case, with a rescaling of coordinates to make $\beta=1$, yields the new solution

$$\begin{aligned}
\rho&=-D\cosh x^1\cos x^2+E\sinh 2x^1\sin 2x^2\\
z+z_0&=E\cosh 2x^1\cos 2x^2+D\sinh x^1\sin x^2\\
\mathrm{e}^H&=Q\cos x^2\operatorname{cosec}^2 x^2, \qquad \mathrm{e}^{\Psi}=D/Q\cosh x^1
\end{aligned} \tag{5.12}$$

where D, E, Q and z_0 are constants. Suitable ranges for the coordinates are $\bar{x}^1<x^1<\infty$, $0<x^2<\pi/2$ for Q, E positive. The value of $\bar{x}^1$ depends on the choice of D and E.

Other solutions for set I in conjunction with the constraints C.I(2) are found by choosing

$$A_1=D_1\,\mathrm{e}^{x^1}. \tag{5.13}$$

This leads to the solutions

$$\begin{aligned}
\rho&=-D\,\mathrm{e}^{x^1}\cos x^2+E\,\mathrm{e}^{x^1/2}\sin(x^2/2)\\
z+z_0&=D\,\mathrm{e}^{x^1}\sin x^2+E\,\mathrm{e}^{x^1/2}\cos(x^2/2)\\
\mathrm{e}^H&=Q/(1-\cos x^2), \qquad \mathrm{e}^{\Psi}=(D/2Q)\,\mathrm{e}^{x^1}
\end{aligned} \tag{5.14}$$

† No solution is possible with $\beta=0$.

and

$$\begin{aligned} \rho &= D\,e^{x^1}\cos x^2 - E\,e^{-2x^1}\sin 2x^2 \\ z+z_0 &= -D\,e^{x^1}\sin x^2 + E\,e^{-2x^1}\cos 2x^2 \\ e^H &= 2Q\cos x^2, \qquad e^{\Psi} = (D/2Q)\,e^{x^1}. \end{aligned} \tag{5.15}$$

Suitable ranges for the coordinates in solution equation (5.14) are $\bar{x}^1 < x^1 < \infty$, $\pi/2 < x^2 < 3\pi/2$ and in equation (5.15), $\bar{x}^1 < x^1 < \infty$, $-\pi/2 < x^2 < \pi/2$ ($E > 0$).

Performing the integrations for set II, or applying the transformation of equations (4.11), we find only one new solution,

$$\begin{aligned} \rho &= D\sinh 2x^1 \sin 2x^2 - E\sinh x^1 \sin x^2 \\ z+z_0 &= D\cosh 2x^1\cos 2x^2 - E\cosh x^1 \cos x^2 \\ e^H &= QE\sin x^2, \qquad e^{\Psi} = (1/Q)(\sinh x^1/\cosh^2 x^1). \end{aligned} \tag{5.16}$$

6. Two special cases

In § 4, we developed the separable functions formalism under the assumption that neither Ψ' nor H' were zero. We now consider the special cases where either one vanishes. From (2.19) and (2.20), we find for $\Psi' = 0$, the following two solutions:

$$\begin{aligned} \rho &= -3k(x^2)^2[\tfrac{1}{3}(x^2)+q] + 2kq^3 + [(x^2)+q][3k(x^1)^2 + k_1x^1 + k_2] \\ z+z_0 &= \tfrac{1}{2}k_1[(x^1)^2 - (x^2)^2 - 2q(x^2)] + k(x^1)[(x^1)^2 - 3(x^2)^2 - 6q(x^2)] + k_2(x^1) \\ \Phi &= kx^1, \qquad e^H = (x^2)+q \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \rho &= [k_1(x^1)+k_2][q-\tfrac{1}{2}(x^2)] - 4k \\ z+z_0 &= -\tfrac{1}{2}k_2(x^1) - \tfrac{1}{4}k_1[4q(x^2) - (x^2)^2 + (x^1)^2] \\ \Phi &= kx^1, \qquad e^H = [q-\tfrac{1}{2}(x^2)]^{-2}. \end{aligned} \tag{6.2}$$

Similarly, when $H' = 0$, there are two solutions,

$$\begin{aligned} \rho &= 3k(x^1)^2[q+\tfrac{1}{3}(x^1)] - 2kq^3 + [(x^1)+q][-3k(x^2)^2 + k_1(x^2) + k_2] \\ z+z_0 &= (x^1)(\tfrac{1}{2}x^1+q)[-6k(x^2)+k_1] + k_2(x^2) + (x^2)^2[k(x^2) - k_1/2] \\ e^{\Psi} &= \Phi' = q+(x^1), \qquad e^H = k \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} \rho &= 4k + (q-\tfrac{1}{2}x^1)[k_1(x^2)+k_2] \\ z+z_0 &= \tfrac{1}{4}k_1[4qx^1 - (x^1)^2 + (x^2)^2] + \tfrac{1}{2}k_2(x^2) \\ \Phi' &= [q-\tfrac{1}{2}(x^1)]^{-2}, \qquad e^H = k \end{aligned} \tag{6.4}$$

where q, k, k_1, k_2 and z_0 are arbitrary constants.

7. Method of complex analytic functions

In § 2, we saw that for $\alpha = 1$, ρ and z satisfy the Cauchy–Riemann equations. Moreover, it is well known from complex analysis that the real and imaginary parts of any complex analytic function also satisfy the Cauchy–Riemann equations. Such functions can be readily constructed and hence we are presented with yet another avenue of approach to new solutions: simply construct an analytic function of the complex variable $x^1+\mathrm{i}x^2$,

$$f=f(x^1+\mathrm{i}x^2).$$

Then choose $\mathrm{Re}(f)=z$, $\mathrm{Im}(f)=\rho$ and a coordinate system which is consistent with the structural constraints is realised. If one can then find a Ψ, H pair (or pairs) which, with the new ρ, satisfy (2.20), a new solution (or solutions) is found. For example, the real and imaginary parts of the complex function

$$f=-\mathrm{i}(x^1+\mathrm{i}x^2)^{-2} \tag{7.1}$$

are

$$\mathrm{Re}(f)=-2x^1x^2[(x^1)^2+(x^2)^2]^{-2}, \tag{7.2}$$
$$\mathrm{Im}\,(f)=[(x^2)^2-(x^1)^2][(x^1)^2+(x^2)^2]^{-2}.$$

Choosing ρ as $\mathrm{Im}\,(f)$ and z as $\mathrm{Re}\,(f)$, we find that

$$\mathrm{e}^{\Psi}=k(x^1)^{-2}, \qquad \mathrm{e}^{H}=k^{-1}(x^2)^{-2} \tag{7.3}$$

satisfy (2.20) and hence we have a new solution. Moreover, this is the first which we have found that does not have ρ as a sum of separable products of functions of x^1 and of x^2 as in (4.1).

8. General formalism and Weyl solutions

There is an additional benefit in modelling the coordinate system to conform to the electrostatic equipotential contours. This derives from the fact that the field equations, re-expressed in this system, readily lend themselves to infinite-series expansions to express all axially symmetric electrostatic fields in a vacuum. We set (Carminati and Cooperstock 1983)

$$\rho\alpha=\sum_j a_j(x^1)b_j(x^2), \qquad \alpha^{-2}=\sum_j A_j(x^1)B_j(x^2). \tag{8.1}$$

Equations (2.13) and (8.1) imply

$$2\,\mathrm{e}^{H}\Phi'=\sum_j b_j\Lambda_j+\sum_{j,k} a_jA_K\,\Omega_{jK}\,; \tag{8.2}$$

$$\Lambda_j\equiv(a_j'+a_j\Phi''/\Phi')', \qquad \Omega_{jk}\equiv[B_k\,\mathrm{e}^{H}(b_j\,\mathrm{e}^{-H})']'.$$

With the power series form $A_j=a_j=(x^1)^j$, equation (8.2) becomes

$$\begin{aligned} 2\,\mathrm{e}^{H}\Phi_n' &= \sum_{\substack{j,k\\ j+k=n}} \Omega_{jk}+\sum_j{}^{*}\, l_j^n b_j,\\ \Phi' &\equiv\sum_n \Phi_n'(x^1)^n, \qquad l_j^n \text{ constants}\end{aligned} \tag{8.3}$$

where Σ^* represents a summation over the linearly independent b_j's.

Now let Λ_j be expressed as a power series

$$\Lambda_j\equiv\sum_k \Lambda_{jk}(x^1)^k. \tag{8.4}$$

It then follows from equations (8.2)–(8.4) that

$$\sum_j b_j \lambda_{jn} = \sum_j{}^{*} l_j^n b_j \tag{8.5}$$

which relates the λ and l coefficients.

At this point, it is of interest to test the entire formalism and prove its utility. To this end, we will now extract the complete set of Weyl solutions. Moreover, we will do so without the use of Weyl's insight that $\int e^{-w}\, d\Phi$ is harmonic. If we demand that the gravitational and electrostatic equipotential contours overlap, then $w = w(x^1)$ since $\Phi = \Phi(x^1)$. It then follows from (2.10) that $\rho\alpha$ must assume the simply separable form $a(x^1)b(x^1)$ and $b e^{-H}$ must be a constant. These results, in conjunction with the formalism, yield

$$(a' + a\Phi''/\Phi')' = 2p\Phi' \tag{8.6}$$

which integrates to

$$\int \frac{dx^1}{a} = \int \frac{d\Phi}{p\Phi^2 + q\Phi + s}, \tag{8.7}$$

where p, q and s are constants.

With (8.7) and the specification of a given coordinate system, the corresponding explicit Weyl solution is immediately found. For example, $x^1 = r$, $x^2 = \cos\theta$ yields the charged Curzon metric (Cooperstock and de la Cruz 1978, 1979).

The fact that the Weyl solutions simply emerge from the $w = w(x^1)$ restriction illustrates the very natural aspect of coordinate modelling. Indeed, the modelling has already incorporated the harmonic aspect through the demand on the constraint for ρ.

9. Transformations and properties of solutions

In § 3, we found for the simply separable case that with $\lambda \neq 0$, the potential Φ could be chosen to approach $\pm(Q^{1/2}/\lambda^2)\sin\theta$ asymptotically. This kind of behaviour lends itself to a transformation from the static electrovac form to a new NUT-like stationary vacuum metric by a Bonnor transformation (Bonnor 1966, 1979). Bonnor has shown that a transformation from the metric of equation (2.1) to the stationary vacuum metric

$$ds^2 = e^{\bar{w}}(dt + \omega\, d\phi)^2 - e^{-\bar{w}}[e^{\bar{v}}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] \tag{9.1}$$

is achieved by taking

$$e^{\bar{w}} = \rho\, e^{-w/2}, \qquad \omega = \Phi \tag{9.2}$$

where ω is the angular velocity. Thus, with the Bonnor transformation, the asymptotic $\sin\theta$ form of Φ is transferred to ω and this is precisely the NUT-like property.

It is also worth noting that Kramer and Neugebauer (1969) and Kinnersley (1973) have developed symmetry transformations which map static solutions into stationary electrovac solutions. For example, the Kinnersley class IV transformation applied to a static electrovac solution yields the stationary electrovac solution where

$$e^{\bar{w}} = e^{w}/(1 + \beta^2\varepsilon^2), \qquad \nabla\times\boldsymbol{\omega} = -2\beta\, e^{-w}\nabla\varepsilon \tag{9.3}$$

$\varepsilon \equiv e^{w} - \Phi^2$, β a real parameter. This, as well as their class V transformation, can be used to generate new stationary metrics from our new static metrics.

From the solutions which we have already found, there is a plethora of possibilities, particularly when one considers all the arbitrary constants. In the following, we enumerate some of the properties of the solutions.

From (3.7), (3.8) and (3.10) for the $\lambda = 0$ case, $\Phi \sim rf(\theta)$ asymptotically where $f(\theta) = (1+3\cos^2\theta)^{1/2}$ or $f(\theta) = \sin\theta$. Thus, the asymptotic form of the electric field is

$$\boldsymbol{E} = -[f(\theta)\hat{r} + f'(\theta)\hat{\theta}] \tag{9.4}$$

which neither diverges nor goes to zero.

From (5.14) and (5.15) with $E = 0$, Φ approaches $r/2Q$ asymptotically and hence the electric field approaches the constant value $-(1/2Q)\hat{r}$.

From (6.4), for k_2 and z_0 set to zero, Φ exhibits line singularities. This is also the case for several of the other possibilities. An outstanding challenge is one of determining non-Weyl solutions by our method which are also asymptotically flat. Although these are the solutions most eagerly sought for their physical interest, other non-asymptotically flat solutions can be of interest (see e.g. Ernst 1976).

10. Concluding remarks

Using the coordinate-modelling technique, we have been able to extract many new non-Weyl solutions of the static axially symmetric electrovac equations. This has been possible because the method forces the mathematical structure to conform to the inherent symmetry of the system. Hopefully, this technique can be used to simplify other problems as well.

Future efforts will be channelled in two directions. Firstly, the inventory of the readily extractable solutions will be filled out. Secondly, it would be most useful if specific physical aspects of a system could be injected at an intermediate step in the formalism. We will attempt this in future work.

Note added in proof. We have now found the most general ρ structure allowed by the $\alpha = 1$ equations (2.19) and (2.20).

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